

Quasi-linear Stokes Phenomenon for the Second Painlevé Transcendent

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Abstract

Using the Riemann-Hilbert approach, we study the quasi-linear Stokes phenomenon for the second Painlevé equation $y_{xx} = 2y^3 + xy - \alpha$. The precise description of the exponentially small jump in the dominant solution approaching α/x as $|x| \rightarrow \infty$ is given. For the asymptotic power expansion of the dominant solution, the coefficient asymptotics is found.

1 Introduction

The classical Painlevé equations [1] are six particular ODEs $y_{xx} = R(x, y, y_x)$ whose general solutions called the Painlevé functions are free from movable critical points and can not be expressed in terms of elementary or classical special functions. The properties of the Painlevé functions show the remarkable similarity to those of the classical transcendents of hypergeometric type like Airy, Bessel, Weber, Whittaker and Gauss functions, see monograph [2] and surveys mentioned there.

The problem of the asymptotic description of the Painlevé functions near fixed critical points is one of the most interesting and important problems in the theory of the Painlevé equations. The first result of such kind was obtained by Boutroux [3], who studied the first and second Painlevé equations, $y_{xx} = 6y^2 + x$ and $y_{xx} = 2y^3 + xy - \alpha$, using direct methods based on the integral estimates. Boutroux has shown that, generically, the asymptotic as $|x| \rightarrow \infty$ behavior of the first and second Painlevé functions is given by the Weierstraß elliptic function and by the elliptic sine, respectively. Periods of these elliptic asymptotic solutions depend on $\arg x$ only. Along the rays $\arg x = \pi + \frac{2\pi}{5}k$ for P1 and $\arg x = \frac{\pi}{3}k$ for P2, $k \in \mathbb{Z}$, the general Painlevé transcendent is described by the degenerate elliptic, i.e. trigonometric, functions. The asymptotic forms of the phase shift in the elliptic ansatz for P1 and P2 as $\arg x$ belongs to the interior of the “elliptic” sectors are found in [4, 5], while the connection formulae for the phase shift for different sectors were obtained in [6, 7]. Similar results for Painlevé equations P4 and P3 were published in [8, 9]. The nontrivial jump in the phase of the elliptic asymptotic ansatz across the rays of the trigonometric asymptotic behavior constitutes the *nonlinear* Stokes phenomenon.

In our terms, the *quasi-linear* Stokes phenomenon is the analog of the classical, or linear, Stokes phenomenon. The latter was observed for the first time in the asymptotic analysis of the Airy equation, $y_{xx} = xy$, and consists in the jump of the recessive, exponentially decreasing solution in the “shadow” of the dominant, exponentially large term.

Boutroux [3] has found that the Painlevé equations admit 1-parameter solutions represented by the sum

$$y = (\text{power series}) + (\text{exponential terms}) \quad (1)$$

of a power series with the leading term $y_0(x)$ which satisfies certain algebraic equation, e.g. $6y_0^2 + x = 0$ for P1 and $2y_0^3 + xy_0 = \alpha$ for P2, and of a *transseries* which is the sum of decreasing exponential terms. The latter can be obtained from the conventional perturbation analysis. Solutions of such kind are usually referred to as the truncated, separatrix, degenerate, quasi-stationary or instanton solutions.

Since all equations of the perturbation analysis are linear homogeneous and non-homogeneous ODEs, there is no surprise to meet the quasi-linear Stokes phenomenon studying the degenerate solutions.

The first, heuristic, description of the quasi-linear Stokes phenomenon was obtained in [10, 11]. There are known several attempts to find the analytic meaning of the quasi-linear Stokes phenomenon and to justify it using the resurgent analysis. The reader can find general settings of the Borel summation technique for the nonlinear differential equations in the paper of Costin [12]. The technique is shown to be effective for certain nonlinear asymptotic problems. For instance, in [13], Costin applied the resurgent approach to find the asymptotic location of the first real pole of the degenerate solution of P1 when the amplitude of its exponentially decreasing perturbation becomes large.

Takei [14] proposed two ways of the use of the Borel summation for description of the quasi-linear Stokes phenomenon in P1. His first idea is similar to one of [12] and is based on the study of the Borel transform of the leading power series. Assuming that the Borel series has certain analytic structure on the boundary of its convergence, Takei obtained a simple formula which relates the jump of the recessive term in (1) to the asymptotics of the Taylor coefficients of the Borel transform. However, the assumptions on the analytic structure of the Borel transform remain unjustified. In the second way, he studied the linear differential equation associated to P1 [15] using the so-called “exact” WKB analysis. This potentially effective approach is based on the assumption of the Borel summability of the formal “instanton” series for the Painlevé function which is still an open problem, see [16].

The coefficient asymptotics in the power series in (1) for the fifth Painlevé equation P5 was found by Basor and Tracy [17] up to a common constant factor. The approximate value for the latter was proposed in [17] using the numerical computations. The exact value of the Basor-Tracy constant as well as the proof

of their asymptotic formula were presented by Andreev and Kitaev in [18] via a combination of the isomonodromy method and the Borel summation technique. We emphasize the importance of the paper [18] as the first exact description of the relation between the coefficient asymptotics in the Painlevé transcendent expansions and the quasi-linear Stokes phenomenon.

In [19], Joshi and Kitaev studied the asymptotics of the leading power series coefficients for the degenerate solution of P1 and announced some results in the asymptotics of the transseries coefficients. The investigation is based on the use of the recursion relations and leaves a common factor for all the asymptotic coefficients undetermined. The authors proposed the value of this factor noting that it could be found using the quasi-linear Stokes phenomenon and the Borel transform technique similar to presented in [18].

The common idea of the approaches mentioned above is the use of the Borel transform, most frequently, to fix the solution having the formal power series as its asymptotics. The problem with this idea is that one needs to assume certain analytic properties of the Borel transform of the solution.

Below, we study the quasi-linear Stokes phenomenon for the second Painlevé equation P2,

$$y_{xx} = 2y^3 + xy - \alpha, \quad \alpha = \text{const}, \quad (2)$$

from the viewpoint of the isomonodromy deformation method [15, 20, 2]. We pursue a two-fold goal, i.e. (a) to make an exact sense of the formal equation (1), and (b) to evaluate the asymptotics of the coefficients of the leading power series in (1).

We stress that we *do not* use the Borel summation at any stage of our investigation. Instead, we fix the solutions of P2 (2) with the power series asymptotic behavior by the proper choice of the monodromy data of the associated linear equation. Unlike [18], our study is based on the direct asymptotic analysis of the Riemann-Hilbert problem via the nonlinear steepest descent method [23]. In this approach, we associate the solution to the RH problem with a disjoint graph. This in turn allows us to separate effectively the exponentially small term from the power-like background, see Section 2. Using the outlined method, we rigorously prove the existence of the solutions $y_1(x)$, $y_2(x)$, $y_3(x)$ asymptotic to α/x as $|x| \rightarrow \infty$ in the respective overlapping sectors $\arg x \in (-\frac{\pi}{3}, \pi)$, $\arg x \in (\frac{\pi}{3}, \frac{5\pi}{3})$, $\arg x \in (-\pi, \frac{\pi}{3})$, and find the exponentially small differences $y_k - y_l$, $k, l = 1, 2, 3$, which constitute the quasi-linear Stokes phenomenon.

The collection of the functions $y_k(x)$, $k = 1, 2, 3$, forms a piece-wise holomorphic function $\hat{y}(x) \sim \frac{\alpha}{x}$ as $|x| \rightarrow \infty$. The moments of this function immediately yield the asymptotics for the coefficients of the leading power series (1).

2 Riemann-Hilbert problem for P2

In [20], Flaschka and Newell formulated the inverse problem method for P2. The method reduces the integration of (2) for $|\operatorname{Re} \alpha| < \frac{1}{2}$ to solution of the matrix

Riemann-Hilbert (RH) problem for an auxiliary function $\Psi(\lambda)$ which in its turn is equivalent to a system of singular integral equations. Using this method, they described the limit of small y and y_x and obtained the determinant formulae for the rational and classical solutions of (2). Later, the RH problem was extensively studied by Fokas, Ablowitz, Deift, Zhou, Its, Kapaev, who considered the questions of existence and uniqueness of the RH problem solution [21, 22], and obtained various asymptotic solutions of the latter [23, 24].

In accord with [20], the Painlevé function set is parameterized by two of the Stokes multipliers of the associated linear system below denoted by the symbols s_k , $k = 1, 2, 3$. We begin with the case of decreasing as $x \rightarrow +\infty$ Painlevé transcendents corresponding to the special value $s_2 = 0$. We observe that, for $0 < \arg x < \frac{\pi}{3}$, the RH problem graph can be transformed to the union of three disjoint branches. Using this fact, we define the dominant term $y_1(x)$ as the solution of the Painlevé equation corresponding to the reduced RH problem with $s_2 = s_3 = 0$. The recessive, exponentially small term appears as the contribution of the disjoint branches in the RH problem graph. Thus the recessive term can be effectively distinguished from the dominant background.

Below, we use the modification of the original RH problem of Flaschka and Newell [20] which is more convenient for our purposes. Also, we restrict ourselves to the case $\alpha - \frac{1}{2} \notin \mathbb{Z}$. The excluded case of the half-integer α can be treated similarly, but requires a separate consideration due to logarithmic behavior of $\Psi(\lambda)$ at the origin.

Let us introduce the piece-wise oriented contour $\gamma = C \cup \rho_+ \cup \rho_- \cup_{k=1}^6 \gamma_k$ which is the union of the rays $\gamma_k = \{\lambda \in \mathbb{C}: |\lambda| > r, \arg \lambda = \frac{\pi}{6} + \frac{\pi}{3}(k-1)\}$, $k = 1, \dots, 6$, oriented to infinity, the clock-wise oriented circle $C = \{\lambda \in \mathbb{C}: |\lambda| = r\}$, and of two vertical radii $\rho_+ = \{\lambda \in \mathbb{C}: |\lambda| < r, \arg \lambda = \frac{\pi}{2}\}$ and $\rho_- = \{\lambda \in \mathbb{C}: |\lambda| < r, \arg \lambda = -\frac{\pi}{2}\}$ oriented to the origin. The contour γ divides the complex λ -plane into 8 regions Ω_k , $k \in \{l, r, 1, \dots, 6\}$. Ω_l and Ω_r are left and right halves of the interior of the circle C deprived the radii ρ_+ , ρ_- . The regions Ω_k , $k = 1, \dots, 6$, are the sectors between the rays γ_k and γ_{k-1} outside the circle.

Let each of the regions Ω_k , $k = r, 6, 1, 2$, be a domain for a holomorphic 2×2 matrix function $\Psi_k(\lambda)$. Denote the collection of $\Psi_k(\lambda)$ by $\Psi(\lambda)$,

$$\Psi(\lambda)|_{\lambda \in \Omega_k} = \Psi_k(\lambda), \quad \Psi(e^{i\pi}\lambda) = \sigma_2 \Psi(\lambda) \sigma_2. \quad (3)$$

Let $\Psi_+(\lambda)$ and $\Psi_-(\lambda)$ be the limits of $\Psi(\lambda)$ on γ to the left and to the right, respectively. Let us also introduce the Pauli matrices $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and two matrices $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The RH problem we talk about is the following one:

- i) Find a piece-wise holomorphic 2×2 matrix function $\Psi(\lambda)$ such that

$$\Psi(\lambda) e^{\theta \sigma_3} \rightarrow I, \quad \lambda \rightarrow \infty, \quad \theta = i\left(\frac{4}{3}\lambda^3 + x\lambda\right), \quad (4)$$

and

$$\|\Psi_r(\lambda)\lambda^{-\alpha\sigma_3}\| \leq \text{const} \quad \text{as } \lambda \rightarrow 0; \quad (5)$$

ii) on the contour γ , the jump condition holds

$$\Psi_+(\lambda) = \Psi_-(\lambda)S(\lambda), \quad (6)$$

where the piece-wise constant matrix $S(\lambda)$ is given by equations:

on the rays γ_k ,

$$S(\lambda)|_{\gamma_k} = S_k, \quad S_{2k-1} = I + s_{2k-1}\sigma_-, \quad S_{2k} = I + s_{2k}\sigma_+, \quad (7)$$

with the constants s_k satisfying the constraints

$$s_{k+3} = -s_k, \quad s_1 - s_2 + s_3 + s_1s_2s_3 = -2\sin\pi\alpha; \quad (8)$$

on the radiuses ρ_\pm , $S(\lambda)$ is specified by the equations

$$\begin{aligned} \lambda \in \rho_-: \quad & \Psi_l(e^{2\pi i}\lambda) = \Psi_r(\lambda)M, \\ \lambda \in \rho_+: \quad & \Psi_r(\lambda) = \Psi_l(\lambda)\sigma_2 M \sigma_2, \end{aligned} \quad (9)$$

where

$$M = ie^{i\pi\alpha\sigma_3}\sigma_1; \quad (10)$$

on the circle C , the piece-wise constant jump matrix $S(\lambda)$ is defined by the equations

$$\begin{aligned} \Psi_6(\lambda) &= \Psi_r(\lambda)ES_6^{-1}, \quad \Psi_1(\lambda) = \Psi_r(\lambda)E, \quad \Psi_2(\lambda) = \Psi_r(\lambda)ES_1, \\ \Psi_3(\lambda) &= \Psi_l(\lambda)\sigma_2 E \sigma_2 S_3^{-1}, \quad \Psi_4(\lambda) = \Psi_l(\lambda)\sigma_2 E \sigma_2, \quad \Psi_5(\lambda) = \Psi_l(\lambda)\sigma_2 E \sigma_2 S_4, \end{aligned} \quad (11)$$

where the unimodular constant matrix E satisfies the equation

$$ES_1S_2S_3 = \sigma_2 M^{-1} E \sigma_2. \quad (12)$$

Because the asymptotics of $\Psi(\lambda)$ as $\lambda \rightarrow \infty$ is given by

$$Y(\lambda) := \Psi(\lambda)e^{\theta\sigma_3} = I + \frac{1}{\lambda}\left(-\frac{iD}{2}\sigma_3 + \frac{y}{2}\sigma_1\right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad D = y_x^2 - y^4 - xy^2 + 2\alpha y, \quad (13)$$

the solution $y(x)$ of the Painlevé equation can be found from the “residue” of $Y(\lambda)$ at infinity,

$$y = 2 \lim_{\lambda \rightarrow \infty} \lambda Y_{12}(\lambda) = 2 \lim_{\lambda \rightarrow \infty} \lambda Y_{21}(\lambda). \quad (14)$$

The equation (14) specifies the Painlevé transcendent as the function $y = f(x, \alpha, \{s_k\})$ of the independent variable x , of the parameter α in the equation and of the Stokes multipliers s_k (in the generic case, the connection matrix

E can be expressed via s_k using the equation (12) modulo the left diagonal multiplier). Using the solution $y = f(x, \alpha, \{s_k\})$ and the symmetries of the Stokes multipliers described in [6], we obtain further solutions of P2:

$$\begin{aligned} y &= -f(x, -\alpha, \{-s_k\}), \quad y = \overline{f(\bar{x}, \bar{\alpha}, \{\overline{s_{4-k}}\})}, \\ y &= e^{i\frac{2\pi}{3}n} f(e^{i\frac{2\pi}{3}n} x, \alpha, \{s_{k+2n}\}), \end{aligned} \quad (15)$$

where the bar means the complex conjugation.

2.1 Asymptotic solution for $s_2 = 0$

Let us consider the RH problem above where $s_2 = 0$ assuming that $|x| \rightarrow \infty$ in the sector $\arg x \in [0, \frac{\pi}{3}]$. Equations (8) imply that $s_5 = 0$ as well, and the function $Y(\lambda)$ defined in (13) has no jump across two rays $\arg \lambda = \frac{\pi}{2}, \frac{3\pi}{2}$, $|\lambda| > r$. The remaining parameters s_1 and s_3 satisfy the constraint

$$s_1 + s_3 = -2 \sin \pi \alpha, \quad (16)$$

and the connection matrix E due to (12) has the form

$$E = \begin{pmatrix} p & \\ & q \end{pmatrix} \begin{pmatrix} 1 & ie^{-i\pi\alpha} \\ 1 & -ie^{i\pi\alpha} \end{pmatrix}, \quad pq = -\frac{1}{2i \cos \pi \alpha}. \quad (17)$$

Our first step in the RH problem analysis is elementary and consists of the formulation of the equivalent RH problem for the piece-wise holomorphic function $Y(\lambda) = \Psi(\lambda)e^{\theta(\lambda)\sigma_3}$:

$$\begin{aligned} i) \quad & Y(\lambda) \rightarrow I \quad \text{as } \lambda \rightarrow \infty, \quad \|Y(\lambda)\lambda^{-\alpha\sigma_3}\| \leq \text{const} \quad \text{as } \lambda \rightarrow 0; \\ ii) \quad & Y_+(\lambda) = Y_-(\lambda)G(\lambda), \quad G(\lambda) = e^{-\theta\sigma_3}S(\lambda)e^{\theta\sigma_3}, \quad \lambda \in \gamma. \end{aligned} \quad (18)$$

If $S(\lambda) = I + s\sigma_{\pm}$ then $G(\lambda) = I + se^{\mp 2\theta}\sigma_{\pm}$. On the second step, we transform the jump contour γ to the contour of the steepest descent for the matrix $G(\lambda) - I$.

Let us denote by γ_+ the level line $\text{Im } \theta(\lambda) = \text{const}$ passing through the stationary phase point $\lambda_+ = \frac{i}{2}x^{1/2}$ and asymptotic to the rays $\arg \lambda = \frac{\pi}{6}, \frac{5\pi}{6}$. This is the steepest descent line for $e^{2\theta}$. Similarly, let us denote by γ_- the level line passing through the stationary phase point $\lambda_- = -\frac{i}{2}x^{1/2}$ and asymptotic to the rays $\arg \lambda = -\frac{\pi}{6}, -\frac{5\pi}{6}$. This is the steepest descent line for $e^{-2\theta}$.

For $\arg x \in (0, \frac{\pi}{3}]$, the level lines ℓ_+ and ℓ_- emanating from the origin and asymptotic to the respective rays $\arg \lambda = \frac{\pi}{6}$ and $\arg \lambda = \frac{5\pi}{6}$ are the steepest descent lines for $e^{2\theta}$ and $e^{-2\theta}$, correspondingly.

For $\arg x = 0$, the level lines ℓ_+ and ℓ_- are the segments connecting the origin $\lambda = 0$ with the respective stationary phase points λ_+ and λ_- . For $x > 4r^2$, these lines contain the radiuses ρ_+ and ρ_- .

It is convenient to consider the following equivalent RH problems for $\Psi(\lambda)$:

for $\arg x \in (0, \frac{\pi}{3}]$, the jump contour is the union of the level lines γ_+ , γ_- , ℓ_+ , ℓ_- oriented from the left to the right, and of the circle $C = C_r \cup C_l$ divided by ℓ_{\pm} into the right C_r and left C_l arcs both clockwise oriented. The jump matrices are as follows:

$$\begin{aligned}
\lambda \in \gamma_+ : S(\lambda) &= S_3^{-1} = \begin{pmatrix} 1 & 0 \\ -s_3 & 1 \end{pmatrix}, \quad \lambda \in \gamma_- : S(\lambda) = S_6 = \begin{pmatrix} 1 & -s_3 \\ 0 & 1 \end{pmatrix}, \\
\lambda \in \ell_+, |\lambda| > r : S(\lambda) &= S_1 S_3 = \begin{pmatrix} 1 & 0 \\ -2 \sin \pi \alpha & 1 \end{pmatrix} = S_+, \\
\lambda \in \ell_-, |\lambda| > r : S(\lambda) &= S_6^{-1} S_4^{-1} = \begin{pmatrix} 1 & -2 \sin \pi \alpha \\ 0 & 1 \end{pmatrix} = S_-, \\
\lambda \in \ell_+, |\lambda| < r : S(\lambda) &= \sigma_2 M^{-1} \sigma_2, \quad \lambda \in \ell_-, |\lambda| < r : S(\lambda) = M, \\
\lambda \in C_r : S(\lambda) &= E, \quad \lambda \in C_l : S(\lambda) = \sigma_2 E \sigma_2.
\end{aligned} \tag{19}$$

The jump contour for the RH problem (19) is decomposed into the disjoint sum of the lines γ_+ , γ_- and $\gamma_0 = \ell_+ \cup \ell_- \cup C$. For the boundary value $\arg x = 0$, the lines ℓ_+ and ℓ_- emanating from the origin pass through the stationary phase points and partially merge with the lines γ_+ and γ_- . Thus the corresponding RH problem graph can not be decomposed into a disjoint union of the steepest descent contours except for $\alpha \in \mathbb{Z}$ when the jump across γ_0 can be eliminated. The particular case $\alpha = 0$ was studied in [20, 2]. The general case is described in the following way:

for $\arg x = 0$, the jump contour is the union of the level lines γ_+ , γ_- oriented from the left to the right, of the level lines ℓ_+ and ℓ_- oriented from λ_- to λ_+ , and of the clockwise oriented circle $C = C_r \cup C_l$. The jump matrices are as follows:

$$\begin{aligned}
\lambda \in \gamma_+ : S(\lambda) &= \begin{cases} S_3^{-1}, & \operatorname{Re} \lambda < 0, \\ S_1, & \operatorname{Re} \lambda > 0, \end{cases} \quad \lambda \in \gamma_- : S(\lambda) = \begin{cases} S_4^{-1}, & \operatorname{Re} \lambda < 0, \\ S_6, & \operatorname{Re} \lambda > 0, \end{cases} \\
\lambda \in \ell_+, |\lambda| > r : S(\lambda) &= S_1 S_3 = \begin{pmatrix} 1 & 0 \\ -2 \sin \pi \alpha & 1 \end{pmatrix}, \\
\lambda \in \ell_-, |\lambda| > r : S(\lambda) &= S_6^{-1} S_4^{-1} = \begin{pmatrix} 1 & -2 \sin \pi \alpha \\ 0 & 1 \end{pmatrix}, \\
\lambda \in \ell_+, |\lambda| < r : S(\lambda) &= \sigma_2 M^{-1} \sigma_2, \quad \lambda \in \ell_-, |\lambda| < r : S(\lambda) = M, \\
\lambda \in C_r : S(\lambda) &= E, \quad \lambda \in C_l : S(\lambda) = \sigma_2 E \sigma_2.
\end{aligned} \tag{20}$$

The problem (20) is the limiting case $\arg x \rightarrow 0$ of (19).

As $\arg x \in [0, \frac{\pi}{3}]$, introduce the reduced RH problem ($s_2 = s_3 = 0$) for the piece-wise holomorphic function $\Phi(\lambda)$ on $\gamma_0 = \ell_+ \cup \ell_- \cup C$ oriented as above:

$$\begin{aligned}
i) \quad & \Phi(\lambda) e^{\theta \sigma_3} \rightarrow I, \quad \lambda \rightarrow \infty, \quad \|\Phi(\lambda) \lambda^{-\alpha \sigma_3}\| \leq \text{const}, \quad \lambda \rightarrow 0, \\
ii) \quad & \Phi_+(\lambda) = \Phi_-(\lambda) S(\lambda), \quad \lambda \in \gamma_0,
\end{aligned} \tag{21}$$

where the jump matrix $S(\lambda)$ is described in (19).

Theorem 2.1 *If $\alpha - \frac{1}{2} \notin \mathbb{Z}$ and $\arg x \in [0, \frac{\pi}{3}]$ while $|x|$ is large enough, then there exists a unique solution of the RH problem (21).*

Proof. Since $\det S(\lambda) \equiv 1$, we have $\det \Phi_+ = \det \Phi_-$, and hence $\det \Phi(\lambda)$ is an entire function. Furthermore, because of the normalization of $\Phi(\lambda)$ at infinity, $\det \Phi(\lambda) \equiv 1$. Let $\tilde{\Phi}$ and Φ be two solutions of (21). Then $\chi(\lambda) = \tilde{\Phi}(\lambda)\Phi^{-1}(\lambda)$ is a rational function of λ with the only possible pole at $\lambda = 0$. However, due to the boundedness of $\Phi(\lambda)\lambda^{-\alpha\sigma_3}$ and $\tilde{\Phi}(\lambda)\lambda^{-\alpha\sigma_3}$ as $\lambda \rightarrow 0$, we have $\chi(\lambda)$ bounded at $\lambda = 0$ and, using the Liouville theorem, $\chi(\lambda) \equiv \text{const}$. Thus, because of the normalization of Φ and $\tilde{\Phi}$ at infinity, $\Phi(\lambda) \equiv \tilde{\Phi}(\lambda)$, and the solution is unique.

To prove the existence of the solution $\Phi(\lambda)$, introduce an auxiliary function

$$\hat{\Phi}^0(z) = B(z) \begin{pmatrix} v_1(z) & v_2(z) \\ v_1'(z) & v_2'(z) \end{pmatrix}, \quad B(z) = \frac{1}{2}e^{-i\frac{\pi}{4}\sigma_3} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\alpha}{z} & 1 \end{pmatrix}, \quad (22)$$

where the prime means differentiation w.r.t. z and, for $\alpha - \frac{1}{2} \notin \mathbb{Z}$,

$$\begin{aligned} v_1(z) &= \sum_{k=0}^{\infty} \frac{\Gamma(\alpha + \frac{1}{2})z^{\alpha+2k}}{4^k k! \Gamma(\alpha + \frac{1}{2} + k)} = 2^{\alpha-\frac{1}{2}} \Gamma(\alpha + \frac{1}{2}) e^{i\frac{\pi}{2}(\alpha-\frac{1}{2})} z^{\frac{1}{2}} J_{\alpha-\frac{1}{2}}(-iz), \\ v_2(z) &= \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2} - \alpha)z^{1-\alpha+2k}}{4^k k! \Gamma(\frac{3}{2} - \alpha + k)} = 2^{\frac{1}{2}-\alpha} \Gamma(\frac{3}{2} - \alpha) e^{i\frac{\pi}{2}(\frac{1}{2}-\alpha)} z^{\frac{1}{2}} J_{\frac{1}{2}-\alpha}(-iz). \end{aligned} \quad (23)$$

Here $J_\nu(z)$ denote the classical Bessel function [25]. It is worth to note that the function $\hat{\Phi}^0(z)$ satisfies the linear differential equation

$$\hat{\Phi}_z^0 = \left(\sigma_3 - \frac{\alpha}{z}\sigma_2\right)\hat{\Phi}^0. \quad (24)$$

As it is easily seen, if $|z| < \text{const}$, then

$$\|\hat{\Phi}^0(z)z^{-\alpha\sigma_3}\| \leq \text{const}. \quad (25)$$

Using the properties of the Bessel functions, we find that the products

$$\hat{\Phi}_1(z) = \hat{\Phi}^0(z)\hat{E}, \quad \hat{\Phi}_2(z) = \hat{\Phi}_1(z)\hat{S}_1, \quad \hat{\Phi}_3(z) = \hat{\Phi}_2(z)\hat{S}_2, \quad (26)$$

where

$$\begin{aligned} \hat{E} &= \frac{\sqrt{\pi}}{2 \cos \pi \alpha} \begin{pmatrix} \frac{2^{1-\alpha}}{\Gamma(\frac{1}{2}+\alpha)} & \\ & \frac{2^\alpha}{\Gamma(\frac{3}{2}-\alpha)} \end{pmatrix} e^{i\frac{\pi}{4}\sigma_3} \begin{pmatrix} e^{-i\pi\alpha} & i \\ ie^{i\pi\alpha} & 1 \end{pmatrix}, \\ \hat{S}_1 &= \begin{pmatrix} 1 & 2 \sin \pi \alpha \\ 0 & 1 \end{pmatrix}, \quad \hat{S}_2 = \begin{pmatrix} 1 & 0 \\ -2 \sin \pi \alpha & 1 \end{pmatrix}, \end{aligned}$$

satisfy the asymptotic relation

$$\hat{\Phi}_k(z) = \left(I - \frac{i\alpha}{2z}\sigma_1 + \mathcal{O}\left(\frac{1}{z^2}\right)\right)e^{z\sigma_3}, \quad z \rightarrow \infty, \quad \arg z \in \left(\pi(k-\frac{3}{2}), \pi(k+\frac{1}{2})\right). \quad (27)$$

We observe also the symmetries,

$$\sigma_2 \hat{\Phi}^0(e^{i\pi} z) \sigma_2 = \hat{\Phi}^0(z) M, \quad M = ie^{i\pi\alpha\sigma_3} \sigma_1, \quad \sigma_2 \hat{\Phi}_{k+1}(e^{i\pi} z) \sigma_2 = \hat{\Phi}_k(z), \quad (28)$$

and the relation

$$\hat{E} \hat{S}_1 = DE, \quad (29)$$

$$D = \frac{\sqrt{\pi} e^{i\frac{\pi}{4}}}{2 \cos \pi\alpha} \begin{pmatrix} \frac{2^{1-\alpha} e^{-i\pi\alpha}}{p\Gamma(\frac{1}{2}+\alpha)} & \\ & \frac{2^\alpha e^{i\pi\alpha}}{q\Gamma(\frac{1}{2}-\alpha)} \end{pmatrix}, \quad E = \begin{pmatrix} p & \\ & q \end{pmatrix} \begin{pmatrix} 1 & ie^{-i\pi\alpha} \\ 1 & -ie^{i\pi\alpha} \end{pmatrix}.$$

Introduce the graph $\hat{\gamma}_0 = \mathbb{R} \cup \hat{C}$ consisting of the real axis oriented from the left to the right and of the clockwise oriented circle $\hat{C} = \{z \in \mathbb{C}: |z| = \hat{r}\}$ divided by the real axis in the lower \hat{C}_d and upper \hat{C}_u arcs. This graph divides the complex z -plane into four regions: $\hat{\Omega}_2$ which is the exterior of the circle in the lower half of the complex z -plane, $\hat{\Omega}_3$ which is the exterior of the circle in the upper half of the complex z -plane, $\hat{\Omega}_d$ which is the lower half of the interior of the circle and $\hat{\Omega}_u$ which is the upper half of the interior of the circle. Define a piece-wise holomorphic function $\hat{\Phi}(z)$,

$$\hat{\Phi}(z) = \begin{cases} \hat{\Phi}_2(z), & z \in \hat{\Omega}_2, \\ \hat{\Phi}_3(z), & z \in \hat{\Omega}_3, \\ \hat{\Phi}^0(z)D, & z \in \hat{\Omega}_d, \\ \sigma_2 \hat{\Phi}^0(e^{-i\pi} z) D \sigma_2, & z \in \hat{\Omega}_u. \end{cases} \quad (30)$$

By construction, this function solves the following RH problem:

$$i) \quad \hat{\Phi}(z) e^{-z\sigma_3} \rightarrow I, \quad z \rightarrow \infty, \quad \|\hat{\Phi}(z) z^{-\alpha\sigma_3}\| \leq \text{const}, \quad z \rightarrow 0; \quad (31)$$

ii) on the contour $\hat{\gamma}_0$, the jump condition $\hat{\Phi}_+(z) = \hat{\Phi}_-(z) \hat{S}(z)$ holds true:

$$\begin{aligned} z > \hat{r}: \quad \hat{S}(z) &= \hat{S}_2 = \begin{pmatrix} 1 & 0 \\ -2 \sin \pi\alpha & 1 \end{pmatrix} = S_+, \\ z < -\hat{r}: \quad \hat{S}(z) &= \hat{S}_1^{-1} = \begin{pmatrix} 1 & -2 \sin \pi\alpha \\ 0 & 1 \end{pmatrix} = S_-, \\ -\hat{r} < z < 0: \quad \hat{S}(z) &= M, \quad 0 < z < \hat{r}: \quad \hat{S}(z) = \sigma_2 M^{-1} \sigma_2, \\ z \in \hat{C}_d: \quad \hat{S}(z) &= E, \quad z \in \hat{C}_u: \quad \hat{S}(z) = \sigma_2 E \sigma_2. \end{aligned} \quad (32)$$

Therefore the function $\hat{\Phi}(z)$ has precisely the jump properties of the function $\Phi(\lambda)$. To find $\Phi(\lambda)$ with the proper asymptotic condition at infinity, let us replace the jump contour $\hat{\gamma}_0$ by the contour obtained from γ_0 using the mapping

$$z(\lambda) = -\theta(\lambda) = -ix\lambda - i\frac{4}{3}\lambda^3. \quad (33)$$

For $|\lambda| \leq R < \frac{1}{2}|x|^{1/2}$, the mapping (33) gives us the holomorphic change of the independent variable. Introduce the piece-wise holomorphic function $\tilde{\Phi}(\lambda)$,

$$\tilde{\Phi}(\lambda) = \begin{cases} \hat{\Phi}(z(\lambda)), & |\lambda| < R, \\ e^{-\theta(\lambda)\sigma_3}, & |\lambda| > R. \end{cases} \quad (34)$$

We look for the solution of the RH problem (21) in the form of the product

$$\Phi(\lambda) = \chi(\lambda)\tilde{\Phi}(\lambda). \quad (35)$$

With this purpose, consider the RH problem for the correction function $\chi(\lambda)$ on the union ℓ of the clockwise oriented circle \mathcal{L} of the radius R divided by the level lines ℓ_+ and ℓ_- into the left \mathcal{L}_l and right \mathcal{L}_r arcs and of the outer parts of the lines ℓ_+ , ℓ_- :

$$\begin{aligned} i) \quad & \chi(\lambda) \rightarrow I, \quad \lambda \rightarrow \infty; \\ ii) \quad & \chi^+(\lambda) = \chi^-(\lambda)H(\lambda), \quad \lambda \in \ell, \\ & \lambda \in \ell_+, |\lambda| > R: \quad H(\lambda) = e^{-\theta\sigma_3}S_+e^{\theta\sigma_3}, \\ & \lambda \in \ell_-, |\lambda| > R: \quad H(\lambda) = e^{-\theta\sigma_3}S_-e^{\theta\sigma_3}, \\ & |\lambda| = R, \lambda \in \mathcal{L}_r: \quad H(\lambda) = \hat{\Phi}_2(z(\lambda))e^{\theta\sigma_3}, \\ & |\lambda| = R, \lambda \in \mathcal{L}_l: \quad H(\lambda) = \hat{\Phi}_3(z(\lambda))e^{\theta\sigma_3}. \end{aligned} \quad (36)$$

Taking into account the equations (32), $S_+ = \hat{S}_2$, $S_- = \hat{S}_1^{-1}$, it is easy to see the continuity of the RH problem at the node points $\ell_+ \cap \mathcal{L}$ and $\ell_- \cap \mathcal{L}$. Furthermore, the jump matrix $H(\lambda)$ satisfies the estimates

$$\|H(\lambda) - I\|, \left\| \frac{\partial H}{\partial \lambda} \right\| \leq \begin{cases} c_1(|x| + |\lambda|^2)e^{-c_2|x||\lambda|}, & \lambda \in \ell_{\pm}, \quad |\lambda| \geq R, \\ c_1R^{-3}, & |\lambda| = R, \end{cases} \quad (38)$$

where the concrete value of the constants c_j , $j = 1, 2$, is not important for us. Because we may take $R = c|x|^{1/2}$ with some positive constant c , the solvability of the RH problem (36) and therefore of (21) is straightforward. Indeed, consider the equivalent system of the non-homogeneous singular integral equations for the limiting value $\chi^-(\lambda)$, i.e.

$$\chi^-(\lambda) = I + \frac{1}{2\pi i} \int_{\ell} \frac{\chi^-(\zeta)(H(\zeta) - I)}{\zeta - \lambda_-} d\zeta, \quad (39)$$

or, in the symbolic form, $\chi^- = I + K\chi^-$. Here λ_- means the right limit of λ on ℓ , and K is the composition of the operator of the right multiplication in $H - I$ and of the Cauchy operator C_- . An equivalent singular integral equation for $\psi^- := \chi^- - I$ differs from (39) in the nonhomogeneous term only,

$$\psi^- = KI + K\psi^-. \quad (40)$$

Consider the integral equation (40) in the space $H^1(\ell)$ of the functions ψ^- such that ψ^- and its distributional derivative both belong to $L^2(\ell)$. Since $H - I$ is small in $H^1(\ell)$ for large enough $|x|$, and C_- is bounded in $H^1(\ell)$ [26], then $\|K\|_{H^1(\ell)} \leq c|x|^{-3/2}$ with some constant c , while $I - K$ is invertible in $H^1(\ell)$. Because $KI \in H^1(\ell)$, equation (40) for ψ^- is solvable in $H^1(\ell)$, and

the solution $\chi(\lambda)$ of the RH problem (36) is determined by $\psi^-(\lambda)$ using the equation $\chi = I + KI + K\psi^-$. \square

In accord with the said above, the function χ^- is given by the converging iterative series, $\chi^- = \sum_{n=0}^{\infty} K^n I$. Thus, for large enough $|x|$ and $\arg x \in [0, \frac{\pi}{3}]$, the Painlevé function $y_1(x)$ corresponding to $s_2 = s_3 = 0$, $s_1 = -2 \sin \pi \alpha$, is given by (14),

$$y_1(x) = 2 \left(\lim_{\lambda \rightarrow \infty} \chi(\lambda) \lambda \right)_{12} = -\frac{1}{\pi i} \int_{\ell} (\chi^-(\zeta)(H(\zeta) - I))_{12} d\zeta. \quad (41)$$

Since $\chi^-(\zeta) = I + \mathcal{O}(\zeta^{-1})$ and $|\zeta| \geq R = c|x|^{1/2}$, we find

$$y_1(x) = -\frac{1}{\pi i} \int_{\ell} (H(\lambda) - I)_{12} d\lambda \cdot (I + \mathcal{O}(x^{-1/2})).$$

Using here the expressions for the jump matrix $H(\lambda)$ from (36), the asymptotics (27), and the definition (33), we find the leading asymptotic term of $y_1(x)$,

$$y_1(x) = \frac{\alpha}{x} + \mathcal{O}(x^{-3/2}), \quad \arg x \in [0, \frac{\pi}{3}]. \quad (42)$$

Let us go to the case of the nontrivial s_3 described by the RH problem (19). We look for the solution $\Psi(\lambda)$ in the form of the product

$$\Psi(\lambda) = X(\lambda)\Phi(\lambda), \quad (43)$$

where $\Phi(\lambda)$ is the solution of the reduced RH problem (21). Thus we arrive at the RH problem for the correction function $X(\lambda)$,

$$\begin{aligned} i) \quad & X(\lambda) \rightarrow I, \quad \lambda \rightarrow \infty, \\ ii) \quad & X_+(\lambda) = X_-(\lambda)\mathcal{H}(\lambda), \quad \lambda \in \gamma_+ \cup \gamma_-, \\ & \lambda \in \gamma_+: \quad \mathcal{H}(\lambda) = \Phi_-(\lambda)S_3^{-1}\Phi_-^{-1}(\lambda), \\ & \lambda \in \gamma_-: \quad \mathcal{H}(\lambda) = \Phi_-(\lambda)S_6\Phi_-^{-1}(\lambda), \end{aligned} \quad (44)$$

where $\Phi_-(\lambda)$ denotes the right limit of $\Phi(\lambda)$ on $\gamma_+ \cup \gamma_-$.

The estimate for the jump matrix on γ_{\pm} ,

$$\|\mathcal{H}(\lambda) - I\| \leq ce^{-\frac{2}{3}|x|^{3/2} \cos(\frac{3}{2} \arg x)} e^{-4|x|^{1/2}|\lambda - \lambda_0|^2}, \quad (45)$$

where c is some constant and $\lambda_0 = \pm \frac{i}{2}x^{1/2}$ is the stationary phase point, yields the estimate for the norm of the singular integral operator \mathcal{K} in the equivalent system of singular integral equations, $X_- = I + \mathcal{K}X_-$,

$$\|\mathcal{K}\|_{L^2(\gamma_+ \cup \gamma_-)} \leq ce^{-\frac{2}{3}|x|^{3/2} \cos(\frac{3}{2} \arg x)}. \quad (46)$$

If $|x|$ is large enough and $\arg x \in [0, \frac{\pi}{3} - \epsilon]$, $\epsilon > 0$, then the operator \mathcal{K} is contracting and the system $X_- = I + \mathcal{K}X_-$ is solvable by iterations in $L^2(\gamma_+ \cup$

γ_-), i.e. $X_- = \sum_{n=0}^{\infty} \mathcal{K}^n X_- = I + \mathcal{K}I + \mathcal{O}(e^{-\frac{4}{3}x^{3/2}})$. However, to incorporate the oscillating direction $\arg x = \frac{\pi}{3}$ in the general scheme, we use some more refined procedure.

Theorem 2.2 *If $s_2 = 0$ and $|x| \rightarrow \infty$, $\arg x \in [0, \frac{\pi}{3}]$, then the asymptotics of the second Painlevé transcendent is given by*

$$y = y_1(x, \alpha) - \frac{is_3}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})) + \mathcal{O}(s_3^2 e^{-\frac{4}{3}x^{3/2}}), \quad (47)$$

where $y_1(x, \alpha) \sim \frac{\alpha}{x}$ is the solution of the Painlevé equation for $s_2 = s_3 = 0$, $s_1 = -2 \sin \pi \alpha$.

Proof. Using the asymptotics of $\Phi(\lambda)$ at infinity (21), we find the asymptotics of the jump matrix $\mathcal{H}(\lambda)$,

$$\mathcal{H}(\lambda) = \begin{cases} I - s_3 e^{2\theta} \sigma_- + \mathcal{O}(\psi^- e^{2\theta}), & \lambda \in \gamma_+, \\ I - s_3 e^{-2\theta} \sigma_+ + \mathcal{O}(\psi^- e^{-2\theta}), & \lambda \in \gamma_-, \end{cases} \quad (48)$$

where $\psi^- = (I - K)^{-1} K I$ introduced above is holomorphic in a neighborhood of γ_{\pm} if $\arg x \in (0, \frac{\pi}{3}]$ and $\psi^- \in H^1(\gamma_+ \cup \gamma_-)$ if $\arg x = 0$ and satisfies the estimate

$$\|\psi^-(\lambda)\| \leq c(|x|^{3/2} + |\lambda|)^{-1}, \quad \lambda \in \gamma_{\pm},$$

with some constant c which value is not important for us.

Consider the triangular matrices $P(\lambda)$, $Q(\lambda)$,

$$\begin{aligned} P(\lambda) &= I - \frac{s_3}{2\pi i} \int_{\gamma_+} e^{2\theta(\zeta)} \frac{d\zeta}{\zeta - \lambda} \sigma_-, \\ Q(\lambda) &= I - \frac{s_3}{2\pi i} \int_{\gamma_-} e^{-2\theta(\zeta)} \frac{d\zeta}{\zeta - \lambda} \sigma_+, \end{aligned} \quad (49)$$

which solve the triangular RH problems:

- i) $P(\lambda) \rightarrow I$ as $\lambda \rightarrow \infty$, and $P_+(\lambda) = P_-(\lambda)(I - s_3 e^{2\theta} \sigma_-)$ as $\lambda \in \gamma_+$,
- ii) $Q(\lambda) \rightarrow I$ as $\lambda \rightarrow \infty$, and $Q_+(\lambda) = Q_-(\lambda)(I - s_3 e^{-2\theta} \sigma_+)$ as $\lambda \in \gamma_-$.

We will look for solution $X(\lambda)$ of the RH problem (44) in the form of the product

$$X(\lambda) = Z(\lambda) Q(\lambda) P(\lambda). \quad (50)$$

The correction function $Z(\lambda)$ satisfies the RH problem

$$\begin{aligned} i) \quad & Z(\lambda) \rightarrow I, \quad \lambda \rightarrow \infty, \\ ii) \quad & Z_+(\lambda) = Z_-(\lambda) V(\lambda), \quad \lambda \in \gamma_+ \cup \gamma_-, \end{aligned} \quad (51)$$

where

$$\begin{aligned} V(\lambda) &= Q(\lambda) P_-(\lambda) \mathcal{H}(\lambda) (I + s_3 e^{2\theta} \sigma_-) P_-^{-1}(\lambda) Q^{-1}(\lambda), \quad \lambda \in \gamma_+, \\ V(\lambda) &= Q_-(\lambda) P(\lambda) \mathcal{H}(\lambda) P^{-1}(\lambda) (I + s_3 e^{-2\theta} \sigma_+) Q_-^{-1}(\lambda), \quad \lambda \in \gamma_-. \end{aligned} \quad (52)$$

Using the first of the relations in (48), we find the estimate for the jump matrix $V(\lambda)$ on γ_+ ,

$$V(\lambda) = I + \mathcal{O}(\psi^- e^{2\theta}) + \mathcal{O}(\phi(\lambda) e^{2\theta}), \quad \lambda \in \gamma_+, \quad (53)$$

where $\phi(\lambda) = \int_{\gamma_-} e^{-2\theta(\zeta)} \frac{d\zeta}{\zeta - \lambda}$ is square integrable and holomorphic in a neighborhood of γ_+ . This function satisfies the estimate

$$|\phi(\lambda)| \leq c(|x|^{1/2} + |\lambda|)^{-1} e^{-\frac{2}{3}|x|^{3/2} \cos(\frac{3}{2} \arg x)}, \quad \lambda \in \gamma_+,$$

with some constant c which value is not important for us. For the product $P\mathcal{H}P^{-1}$ on γ_- , we compute the expression

$$P(\lambda)\mathcal{H}(\lambda)P^{-1}(\lambda) = I - s_3 e^{-2\theta} \sigma_+ + \mathcal{O}(\varphi(\lambda) e^{-2\theta}) + \mathcal{O}(\psi^- e^{-2\theta}), \quad \lambda \in \gamma_-,$$

where $\varphi(\lambda) = \int_{\gamma_+} e^{2\theta(\zeta)} \frac{d\zeta}{\zeta - \lambda}$ is square integrable and holomorphic in a neighborhood of γ_- . This function satisfies the estimate

$$|\varphi(\lambda)| \leq c(|x|^{1/2} + |\lambda|)^{-1} e^{-\frac{2}{3}|x|^{3/2} \cos(\frac{3}{2} \arg x)}, \quad \lambda \in \gamma_-,$$

with some constant c . Thus,

$$V(\lambda) = I + \mathcal{O}(\psi^- e^{-2\theta}) + \mathcal{O}(\varphi e^{-2\theta}), \quad \lambda \in \gamma_-. \quad (54)$$

Our next steps are similar to presented in the proof of Theorem 2.1. Consider the system of the singular integral equations for $Z_-(\lambda)$ equivalent to the RH problem (51), $Z_- = I + KZ_-$. Here the singular integral operator K is the superposition of the multiplication operator in $V - I$ and of the Cauchy operator C_- . Because the Cauchy operator is bounded in $L^2(\gamma_+ \cup \gamma_-)$, the singular integral operator K for large enough $|x|$, $\arg x \in [0, \frac{\pi}{3}]$, satisfies the estimate

$$\|K\|_{L^2(\gamma_+ \cup \gamma_-)} \leq c|x|^{-1/2} e^{-\frac{2}{3}|x|^{3/2} \cos(\frac{3}{2} \arg x)}. \quad (55)$$

Thus equation $\zeta_- = KI + K\zeta_-$ for the difference $\zeta_- := Z_- - I$ is solvable by iterations in the space $L^2(\gamma_+ \cup \gamma_-)$. Solution of the RH problem (51) is given by the integral $Z = I + KI + K\zeta_-$. Thus, using (14), (43), (50) and the asymptotics of P, Q from (49), we arrive at (47). \square

2.2 The decreasing degenerate Painlevé functions

Applying the second of the symmetries (15) to (47), we obtain

Theorem 2.3 *If $s_2 = 0$ and $|x| \rightarrow \infty$, $\arg x \in [-\frac{\pi}{3}, 0]$, then the asymptotics of the second Painlevé transcendent is given by*

$$y = y_3(x, \alpha) + \frac{is_1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})) + \mathcal{O}(s_1^2 e^{-\frac{4}{3}x^{3/2}}), \quad (56)$$

where $y_3(x, \alpha) = \overline{y_1(\bar{x}, \bar{\alpha})} \sim \frac{\alpha}{x}$ is the solution of the Painlevé equation for $s_2 = s_1 = 0$, $s_3 = -2 \sin \pi \alpha$.

The solutions $y_1(x, \alpha)$ and $y_3(x, \alpha) = \overline{y_1(\bar{x}, \bar{\alpha})}$ are meromorphic functions of $x \in \mathbb{C}$ and thus can be continued beyond the sectors indicated in Theorems 2.2 and 2.3. To find the asymptotics of the solution $y_3(x, \alpha)$ in the interior of the sector $\arg x \in [0, \frac{\pi}{3}]$, we apply (47). Similarly, we find the asymptotics of the solution $y_1(x, \alpha)$ in the interior of the sector $\arg x \in [-\frac{\pi}{3}, 0]$ using (56). Both the expressions imply that, if $|x| \rightarrow \infty$, $\arg x \in [-\frac{\pi}{3}, \frac{\pi}{3}]$,

$$y_3(x, \alpha) - y_1(x, \alpha) = i \frac{\sin \pi \alpha}{\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})) + \mathcal{O}(e^{-\frac{4}{3}x^{3/2}}). \quad (57)$$

Remark 1. The same asymptotic relation can be obtained using the analysis of the RH problem for the ratio $\hat{\Psi}(\lambda)\Psi^{-1}(\lambda)$ where $\hat{\Psi}(\lambda)$ is the solution of the RH problem for $s_2 = s_1 = 0$ while $\Psi(\lambda)$ is the solution for $s_2 = s_3 = 0$.

Besides $y_1(x, \alpha)$ and $y_3(x, \alpha)$, let us introduce the solution $y_2(x, \alpha)$ corresponding to the Stokes multipliers $s_1 = \underline{s_3} = 0$, $s_2 = 2 \sin \pi \alpha$. Due to the second of the symmetries (15), $y_2(x, \alpha) = y_2(\bar{x}, \bar{\alpha})$.

The last of the symmetries (15) yields the relations

$$\begin{aligned} y_1(x, \alpha) &= e^{i\frac{2\pi}{3}} y_2(e^{i\frac{2\pi}{3}} x, \alpha) = e^{-i\frac{2\pi}{3}} y_3(e^{-i\frac{2\pi}{3}} x, \alpha), \\ y_2(x, \alpha) &= e^{i\frac{2\pi}{3}} y_3(e^{i\frac{2\pi}{3}} x, \alpha) = e^{-i\frac{2\pi}{3}} y_1(e^{-i\frac{2\pi}{3}} x, \alpha), \\ y_3(x, \alpha) &= e^{i\frac{2\pi}{3}} y_1(e^{i\frac{2\pi}{3}} x, \alpha) = e^{-i\frac{2\pi}{3}} y_2(e^{-i\frac{2\pi}{3}} x, \alpha), \end{aligned} \quad (58)$$

which imply the decreasing asymptotics of the functions $y_k(x, \alpha)$ in the following sectors:

$$\begin{aligned} y_1(x, \alpha) &\sim \frac{\alpha}{x}, \quad |x| \rightarrow \infty, \quad \arg x \in [0, \frac{2\pi}{3}], \\ y_2(x, \alpha) &\sim \frac{\alpha}{x}, \quad |x| \rightarrow \infty, \quad \arg x \in [\frac{2\pi}{3}, \frac{4\pi}{3}], \\ y_3(x, \alpha) &\sim \frac{\alpha}{x}, \quad |x| \rightarrow \infty, \quad \arg x \in [-\frac{2\pi}{3}, 0]. \end{aligned} \quad (59)$$

The same symmetry (15) applied to the equation (57) yields the differences

$$\begin{aligned} \arg x \in [-\pi, -\frac{\pi}{3}]: \\ y_2(x, \alpha) - y_3(x, \alpha) &= -\frac{\sin \pi \alpha}{\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})) + \mathcal{O}(e^{\frac{4}{3}x^{3/2}}), \\ \arg x \in [\frac{\pi}{3}, \pi]: \\ y_1(x, \alpha) - y_2(x, \alpha) &= \frac{\sin \pi \alpha}{\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})) + \mathcal{O}(e^{\frac{4}{3}x^{3/2}}). \end{aligned} \quad (60)$$

Equations (57), (60) constitute the quasi-linear Stokes phenomenon for the second Painlevé equation.

Finally, applying the last of the symmetries (15) to (47), (56), and using (58) we find the asymptotics of the degenerate Painlevé functions:

i) if $s_1 = 0$, then $s_3 - s_2 = -2 \sin \pi \alpha$, and

$$\begin{aligned} \arg x \in [-\pi, -\frac{2\pi}{3}]: y &= y_2(x, \alpha) - \frac{s_3}{2\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})), \\ \arg x \in [-\frac{2\pi}{3}, -\frac{\pi}{3}]: y &= y_3(x, \alpha) - \frac{s_2}{2\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})); \end{aligned} \quad (61)$$

ii) if $s_2 = 0$, then $s_1 + s_3 = -2 \sin \pi \alpha$ and

$$\begin{aligned} \arg x \in [-\frac{\pi}{3}, 0]: y &= y_3(x, \alpha) + \frac{is_1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})), \\ \arg x \in [0, \frac{\pi}{3}]: y &= y_1(x, \alpha) - \frac{is_3}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})); \end{aligned} \quad (62)$$

iii) if $s_3 = 0$, then $s_1 - s_2 = -2 \sin \pi \alpha$ and

$$\begin{aligned} \arg x \in [\frac{\pi}{3}, \frac{2\pi}{3}]: y &= y_1(x, \alpha) - \frac{s_2}{2\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})), \\ \arg x \in [\frac{2\pi}{3}, \pi]: y &= y_2(x, \alpha) - \frac{s_1}{2\sqrt{\pi}} x^{-1/4} e^{\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})). \end{aligned} \quad (63)$$

3 The power expansion of the degenerate solutions and the coefficient asymptotics

Using the steepest descent approach, cf. [27], we can show the existence of the asymptotic expansion of $y_k(x, \alpha)$, $k = 1, 2, 3$, in the negative degrees of $x^{1/2}$. Further elementary investigation of the recursion relation for the coefficients of the series allows us to claim that the asymptotic expansion for any of the decreasing degenerate solutions has the following form:

$$y_0(x, \alpha) = \frac{\alpha}{x} \sum_{n=0}^{\infty} a_n x^{-3n} + \mathcal{O}(x^{-\infty}), \quad (64)$$

where coefficients a_n are determined uniquely by the recurrence relation

$$a_0 = 1, \quad a_{n+1} = (3n+1)(3n+2)a_n - 2\alpha^2 \sum_{\substack{k,l,m=0 \\ k+l+m=n}}^n a_k a_l a_m. \quad (65)$$

The initial terms of the expansion are given by

$$y_0(x, \alpha) = \frac{\alpha}{x} \left\{ 1 + \frac{2(1-\alpha^2)}{x^3} + \frac{4(10-13\alpha^2+3\alpha^4)}{x^6} + \right. \quad (66)$$

$$\begin{aligned}
& + \frac{8}{x^9} (280 - 397\alpha^2 + 129\alpha^4 - 12\alpha^6) + \\
& + \frac{16}{x^{12}} (15400 - 22736\alpha^2 + 8427\alpha^4 - 1146\alpha^6 + 55\alpha^8) + \mathcal{O}(x^{-15}) \Big\}.
\end{aligned}$$

For $\alpha = 0$, the recurrence (65) is exactly solvable,

$$a_n = \frac{\Gamma(3n)}{3^{n-1}\Gamma(n)}, \quad \alpha = 0. \quad (67)$$

Our next goal is to determine the asymptotics of the coefficients a_n in (64) as $n \rightarrow \infty$ for arbitrary α . With this purpose, let us construct a sectorial analytic function $\hat{y}(x)$,

$$\hat{y}(x) = \begin{cases} y_3(x, \alpha), & \arg x \in (-\frac{2\pi}{3}, 0), \\ y_1(x, \alpha), & \arg x \in (0, \frac{2\pi}{3}), \\ y_2(x, \alpha), & \arg x \in (\frac{2\pi}{3}, \frac{4\pi}{3}). \end{cases} \quad (68)$$

The function $\hat{y}(x)$ has a finite number of simple poles. Therefore $\hat{y}(x)$ is bounded for $|x| \geq \rho$ and has the uniform expansion (64) near infinity. Let $y^{(N)}(x)$ be a partial sum

$$y^{(N)}(x, \alpha) = \frac{\alpha}{x} \sum_{n=0}^{N-1} a_n x^{-3n}, \quad (69)$$

and $v^{(N)}(x)$ be a product

$$v^{(N)}(x) = x^{3N}(\hat{y}(x) - y^{(N)}(x, \alpha)) = \frac{\alpha}{x} \sum_{n=0}^{\infty} a_{n+N} x^{-3n} + \mathcal{O}(x^{-\infty}). \quad (70)$$

Because $x^{3N}y^{(N)}(x, \alpha)$ is polynomial, the integral of $v^{(N)}(x)$ along the circle of the radius $|x| = \rho$ satisfies the estimate

$$\left| \oint_{|x|=\rho} v^{(N)}(x) dx \right| \leq \rho^{3N} \oint_{|x|=\rho} |\hat{y}(x)| dl \leq 2\pi \rho^{3N+1} \max_{|x|=\rho} |\hat{y}(x)| = C \rho^{3N+1}. \quad (71)$$

On the other hand, inflating the sectorial arcs of the circle $|x| = \rho$, we find that

$$\begin{aligned}
& \oint_{|x|=\rho} v^{(N)}(x) dx = \oint_{|x|=R} v^{(N)}(x) dx + \\
& + \int_{\rho}^R x^{3N} (y_1 - y_3) dx + \int_{e^{i\frac{2\pi}{3}}\rho}^{e^{i\frac{2\pi}{3}}R} x^{3N} (y_2 - y_1) dx + \int_{e^{-i\frac{2\pi}{3}}\rho}^{e^{-i\frac{2\pi}{3}}R} x^{3N} (y_3 - y_2) dx. \quad (72)
\end{aligned}$$

Because $v^{(N)}(x) = \frac{\alpha}{x} a_N + \mathcal{O}(x^{-4})$, the first of the integrals in the r.h.s. of (72) is computed as follows:

$$\oint_{|x|=R} v^{(N)}(x) dx = 2\pi i \alpha a_N + \mathcal{O}(R^{-3}). \quad (73)$$

Last three integrals in (72) are computed using (57), (60), (58):

$$\begin{aligned}
& \int_{\rho}^R x^{3N} (y_1 - y_3) dx + \int_{e^{i\frac{2\pi}{3}}\rho}^{e^{i\frac{2\pi}{3}}R} x^{3N} (y_2 - y_1) dx + \int_{e^{-i\frac{2\pi}{3}}\rho}^{e^{-i\frac{2\pi}{3}}R} x^{3N} (y_3 - y_2) dx = \\
& = 3 \int_{\rho}^R x^{3N} (y_1 - y_3) dx = -3i \frac{\sin \pi \alpha}{\sqrt{\pi}} \int_{\rho}^R x^{3N-\frac{1}{4}} e^{-\frac{2}{3}x^{3/2}} (1 + \mathcal{O}(x^{-1/2})) dx = \\
& = -3i \frac{\sin \pi \alpha}{\sqrt{\pi}} \left(\frac{3}{2}\right)^{2N-\frac{1}{2}} \left[\Gamma(2N + \frac{1}{2}) (1 + \mathcal{O}(N^{-1/3})) + \mathcal{O}(\rho^{3N}) + \mathcal{O}(R^{3N} e^{-\frac{2}{3}R^{3/2}}) \right].
\end{aligned}$$

Thus, letting $R = \infty$, we find that the asymptotics as $N \rightarrow \infty$ of the coefficient a_N in (64) is given by

$$a_N = \frac{\sin \pi \alpha}{\alpha \pi^{3/2}} (3/2)^{2N+\frac{1}{2}} \Gamma(2N + \frac{1}{2}) (1 + \mathcal{O}(N^{-1/3})) + \mathcal{O}(C^N), \quad N \rightarrow \infty. \quad (74)$$

For $\alpha = 0$, this equation is consistent with equation (67). For $\alpha \in \mathbb{Z} \setminus \{0\}$, the asymptotics (74) is reduced to $a_N = \mathcal{O}(C^N)$ which is consistent with the expansion of a rational solution of P2 determined by the triviality condition $s_1 = s_2 = s_3 = 0$ and $\alpha \in \mathbb{Z}$.

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